1. Introduction

1.1. The general Constraint Satisfaction Problem (CSP)

Many real world problems, such as resource allocation, temporal reasoning, scheduling or scene labelling, naturally appear as Constraint Satisfaction Problems (CSP) [Guesguen et al. 1992, Tsang 1993]. Many theoretical problems and many logic games are also natural examples of CSPs: graph colouring, graph matching, cryptarithmetic, n-Queens, Latin Squares, Sudoku and its innumerable variants… In the past decades, the study of such problems has evolved into a main sub-area of Artificial Intelligence (AI) with its own specialised techniques.

1.1.1. Statement of the Constraint Satisfaction Problem

A CSP is defined by:

– a set of variables X_1, X_2, … X_n, the "CSP variables", each with values in a given domain $Dom(X_1)$, $Dom(X_2)$, … $Dom(X_n)$,

– a set of constraints (i.e. of relations) these variables must satisfy.

The problem consists of finding a value in the domain of each of these variables, such that these values satisfy all the constraints. Later (in Chapter 3), we shall show that a CSP can be re-written as a theory in First Order Logic.

As in many studies of CSPs, all the CSPs we shall consider in this book will be finite, i.e. the number of variables, each of their domains and the number of constraints will all be finite. When we write "CSP", it should therefore always be read as "finite CSP".

Also, we shall consider only CSPs with binary constraints. One can always tackle unary constraints by an appropriate choice of the domains. And, for $k > 2$, a k-ary constraint between a subset of k variables (X_{n1}, .., X_{nk}) can always be replaced by k binary constraints between each of these X_{ni} and an additional variable representing the original k-ary constraint; although this new variable has a large domain and this may not be the most efficient way of dealing with the given k-ary constraint, this is a very standard approach (for details, see [Tsang 1993]).

Moreover, a binary CSP can always be represented as a (generally large) labelled undirected graph: a node of this graph, called a *label*, is a possible value of some of the CSP variables (or, as we shall see, an equivalence class of such possibilities); given two nodes in this graph, each binary constraint not satisfied by this pair of labels (including the "strong" constraints induced by CSP variables, i.e. all the contradictions between different values for the same CSP variable) gives rise to an arc between them, labelled by the name of the constraint and representing it. We shall call this graph *the CSP graph*. (Notice that this is different from what is usually called the constraint graph.) The CSP graph expresses all the direct contradictions between any two labels (whereas the constraint graph usually considered expresses their compatibilities).

1.1.2. The Sudoku example

As explained in the foreword, Sudoku has been at the origin of our work on CSPs. In this book we shall keep it as our main example, even though we shall also refer to other CSPs in order to palliate its specificities.

Let us start with the usual formulation of the problem (with its specific vocabulary in italics): given a 9×9 *grid*, partially filled with *numbers* from 1 to 9 (the *givens* of the problem, also called the *clues* or the *entries*), *complete* it with numbers from 1 to 9 in such a way that in each of the nine *rows*, in each of the nine *columns* and in each of the nine disjoint *blocks* of 3×3 contiguous *cells*, the following property holds:

– there is at most one occurrence of each of these numbers.

Although this defining condition could be replaced by either of the following two, which are obviously equivalent to it, we shall stick to the first formulation, for reasons that will appear later:

– there is at least one occurrence of each of these numbers,

– there is exactly one occurrence of each of these numbers.

Figure 1.1 shows the standard presentations of a *problem grid* (also called a *puzzle*) and of a *solution grid* (also called a *complete Sudoku grid*).

							1	2
				3	5			
			6				7	
7						3		
			4			8		
1								
			1	2				
	8						4	
	5					6		

6	7	3	8	9	4	5	1	2
9	1	2	7	3	5	4	8	6
8	4	5	6	1	2	9	7	3
7	9	8	2	6	1	3	5	4
5	2	6	4	7	3	8	9	1
1	3	4	5	8	9	2	6	7
4	6	9	1	2	8	7	3	5
2	8	7	3	5	6	1	4	9
3	5	1	9	4	7	6	2	8

Figure 1.1. *A puzzle (Royle17#3) and its solution*

Since rows, columns and blocks play similar roles in the defining constraints, they will naturally appear to do so in many other places and a word that makes no difference between them is widely used in the Sudoku world: a *unit* is either a row or a column or a block. And one says that two cells *share a unit*, or that they *see* each other, if they are different and they are either in the same row or in the same column or in the same block (where "or" is non exclusive). We shall also say that these two cells are *linked*. It should be noticed that this (symmetric) relation between two different cells, whichever of the three equivalent names it is given, does not depend on the content of these cells but only on their place in the grid; it is therefore a straightforward and quasi physical notion.

As appears from the definition, a Sudoku grid is a special case of a Latin Square. Latin Squares must satisfy the same constraints as Sudoku, except the condition on blocks. The practical consequences of this relationship between Sudoku and Latin Squares will appear throughout this book and, following *HLS1*, the logical relationship between the two theories will be fully clarified in chapters 3 and 4.

What we need now is to see how the above natural language formulation of the Sudoku problem can be re-written as a CSP. In Chapter 2, the essential question of modelling in general and its practical implications on how to deal with a CSP will be raised and we shall see that the following formalisation is neither the only one nor the best one. But, for the time being, we only want to write the most straightforward one.

For each row r and each column c, introduce a variable X_{rc}, with domain the set {1, 2, 3, 4, 5, 6, 7, 8, 9}. Then the general Sudoku problem can be expressed as a CSP with the following set of (binary) constraints:
$X_{rc} \neq X_{r'c'}$ for all the pairs {rc, r'c'} such that the cells rc and r'c' share a unit,
and a particular puzzle will add to this the set of unary constraints fixing the values of the X_{rc} corresponding to the givens.

Notice that the natural language phrase "complete the grid" in the original formulation has naturally been understood as "assign one and only one value to each of the cells" – which has then been translated into "assign a value to each of the X_{rc} variables" in the CSP formulation.

1.2. Paradigms of resolution

A CSP states the constraints a solution must satisfy, i.e. it says *what* is desired. But it does not say anything about *how* a solution can be obtained; this is the job of resolution methods, the choice of which will depend on the various purposes one may have in addition to merely finding a solution. A particular class of resolution methods, based on *resolution rules*, will be the main topic of this book.

1.2.1. Various purposes and methods

If the only goal is to get a solution by any available means, very efficient general-purpose algorithms have been known for a long time [Kumar 1992, Tsang 1993]; they guarantee that they will either find one solution or all the solutions (according to what is desired) or find a contradiction in the givens; they have lots of more recent variants and refinements. Most of these algorithms involve the combination of two very different techniques: some direct propagation of constraints between variables (in order to progressively reduce their sets of possible values) and some kind of structured search with "backtracking" (depth-first, breadth-first, …, possibly with some forms of look-ahead); they consist of trying (recursively if necessary) a value for a variable and propagating (based on the constraints) the consequences of this tentative choice as restrictions on other variables; eventually, either a solution or a contradiction will be reached; the latter case allows to conclude that this value (or this combination of values simultaneously tried in the recursive case) is impossible and it restricts the possibilities for this (subset of) variables(s).

But, in some cases, such blind search is not possible for practical reasons (e.g. one is not in a simulator but in real life) or not allowed (for *a priori* theoretical or æsthetic reasons), or one wants to simulate human behaviour, or one wants to "understand" each step of the resolution process (as is the case with most Sudoku players), or one wants a "pure logic" or a "constructive" or the "simplest" solution, whatever meaning can be ascribed to the quoted words.

Contrary to the current CSP literature, this book will only be concerned with the latter cases, in which more attention is paid to the resolution path than to the final solution itself. Indeed, it can also be considered as an informal reflection on how notions such as "understandable proof of the solution", "pure logic", "constructive" and "simplest" can be defined (but we shall only be able to say more on this topic in the retrospective "final remarks" chapter). It does not mean that efficiency questions are not relevant to our approach, but they are not our primary goal, they are conditioned by such higher-level requirements. Without these additional requirements, there is no reason to use techniques computationally much harder (probably exponentially much harder) than the general-purpose algorithms.

In such situations, it is convenient to introduce the notion of a *candidate*, i.e. of a "still possible" value for a variable. As this intuitive notion does not pertain to the CSP problem itself, it must first be given a clear definition and a logical status. When this is done (in chapter 4), one can define the concepts of a *resolution rule* (a logical formula in the "condition ⇒ action" form, which says what to do in some factual, observable situation described by the condition pattern), a *resolution theory* (a set of such rules), a *resolution strategy* (a particular way of using the rules in a resolution theory). One can then study the relationship between the original CSP and several of its resolution theories. One can also introduce several properties a resolution theory can have, such as confluence and completeness (contrary to general purpose algorithms, a resolution theory cannot in general solve all the instances of a given CSP; evaluating its scope is thus a new topic in its own; one can also study its statistical resolution power in specific examples). This "pattern-based" approach was first introduced in *HLS1*, in the limited context of Sudoku. It is the purpose of this book to show that it is indeed very general, but let us first illustrate how these ideas work concretely for Sudoku.

1.2.2. Candidates and candidate elimination in Sudoku

The process of solving a Sudoku puzzle "by hand" is generally initialised by defining the "candidates" for each cell. For later formalisation, one must be careful with this notion: if one analyses the natural way of using it, it appears that, *at any stage of the resolution process, a candidate for a cell is a number that has not yet been explicitly proven to be an impossible value for this cell.*

	c1	c2	c3	c4	c5	c6	c7	c8	c9	
r1	3 4 5 6 8 9	3 4 6 7 9	3 4 5 6 7 8 9	7 8 9	4 7 8 9	4 7 8 9	4 5 9	1	2	r1
r2	2 4 6 8 9	1 2 4 6 7 9	1 2 4 6 7 8 9	2 7 8 9	3	5	4 9	6 8 9	4 6 8 9	r2
r3	2 3 4 5 8 9	1 2 3 4 9	1 2 3 4 5 8 9	6	1 4 8 9	1 2 4 8 9	4 5 9	7	3 4 5 8 9	r3
r4	7	2 4 6 9	2 4 5 6 8 9	2 5 8 9	1 5 6 8 9	1 2 6 8 9	3	2 5 6 9	1 4 5 6 9	r4
r5	2 3 5 6 9	2 3 6 9	2 3 5 6 9	4	1 5 6 7 9	1 2 3 6 7 9	8	2 5 6 9	1 5 6 7 9	r5
r6	1	2 3 4 6 9	2 3 4 5 6 8 9	2 3 5 7 8 9	5 6 7 8 9	2 3 6 7 8 9	2 4 5 7 9	2 5 6 9	4 5 6 7 9	r6
r7	3 4 6 9	3 4 6 7 9	3 4 6 7 9	1	2	3 4 6 7 8 9	5 7 9	3 5 8 9	3 5 7 8 9	r7
r8	2 3 6 9	8	1 2 3 6 7 9	3 5 7 9	5 6 7 9	3 6 7 9	1 2 5 7 9	4	1 3 5 7 9	r8
r9	2 3 4 9	5	1 2 3 4 7 9	3 7 8 9	4 7 8 9	3 4 7 8 9	6	2 3 8 9	1 3 7 8 9	r9
	c1	c2	c3	c4	c5	c6	c7	c8	c9	

Figure 1.2. *Grid Royle17#3 of Figure 1.1, with the candidates remaining after the elementary constraints for the givens have been propagated*

Usually, candidates for a cell are displayed in the grid as smaller and/or clearer digits in this cell (as in Figure 1.2). Similarly, at any stage, a *decided value* is a number that has been explicitly proven to be the only possible value for this cell; it is written in big fonts, like the givens.

At the start of the game, one possibility is to consider that any cell with no input value admits all the numbers from 1 to 9 as candidates – but more subtle initialisations are possible (e.g. as shown in Figure 1.2) and a slightly different, more symmetric, view of candidates can be introduced (see chapter 2).

Then, according to the formalisation introduced in *HLS1*, a resolution process that corresponds to the vague requirement of a "pure logic" solution is a sequence of steps consisting of repeatedly applying "resolution rules" of the general condition-action type: if some *pattern* – i.e. configuration of cells, possible cell-values, links, decided values and candidates – defined by the condition part of the rule, is effectively present in the grid, then carry out the action(s) specified by the action part of the rule. Notice that any such pattern always has a purely "physical", invariant part (which may be called its "physical" or structural support), defined by conditions on possible cell-values and on links between them, and an additional part, related to the actual presence/absence of decided values and/or candidates in these cells in the current situation. (Again, this will be generalised in chapter 2 with the four "2D" views.)

Depending on the type of their action part, such resolution rules can be classified into two categories:

– either they assert a decided value for a cell (e.g. when it is proven that there is only one possibility left for it); there are very few rules of this type;

– or they eliminate some candidate(s) (which we call the *target(s)* of the pattern); as appears from a quick browsing of the available literature, almost all the classical Sudoku resolution rules are of this type; they express specific elaborated forms of constraints propagation; their general form is: if such pattern is present, then it is impossible for some number(s) to be in some cell(s) and the corresponding candidates must therefore be deleted; for the general CSP, it will appear that almost all the rules we shall meet in this book are also of this type.

The interpretation of the above resolution rules, whatever their type, should be clear: none of them claims that there is a solution with this value asserted or with this candidate deleted. Rather, it must be interpreted as saying: "from the current situation it can be asserted that any solution, if there is any, must satisfy the conclusion of this rule".

From both theoretical and practical points of view, it is also important to notice that, as one proceeds with resolution, candidates form a monotone decreasing set and decided values form a monotone increasing set. Whereas the notion of a candidate is the intuitive one for players, what is classical in logic is increasing monotonicity (what is known / what has been proven can only increase with time); but this is not a real problem, as it could easily be restored by considering non-candidates instead (i.e. what has been erased instead of what is still present).

For some very difficult puzzles, it seems necessary to (recursively) make a hypothesis on the value of a cell, to analyse its consequences and to eliminate it if it leads to a contradiction; techniques of this kind do not fit *a priori* the above condition-action form; they are proscribed by purists (for the main reason that they often make the game totally uninteresting) and they are assigned the infamous, though undefined, name of Trial-and-Error. As shown in *HLS* and in the statistics of chapter 6, they are needed in only extremely rare cases if one admits the kinds of chain rules (whips) that will be introduced in chapter 5.

1.2.3. Extension of this model of resolution to the general CSP

It appears that the above ideas can be generalised from Sudoku to any CSP. Candidate elimination corresponds to the now classical idea of domain restriction in CSPs. What has been called a candidate above is related to the notion of a *label* in the CSP world, a name coming from the domain of scene labelling, historically at the origin of the identification of the general Constraint Satisfaction Problem. However, contrary to labels that can be given a very simple set theoretic definition based on the data defining the CSP, the status of a candidate is *a priori* not clear from the point of view of mathematical logic, because this notion does not pertain *per se* to the CSP formulation, nor to its direct logic transcription.

In chapter 4, we shall show that a formal definition of a candidate must rely on intuitionistic logic and we shall introduce more formally our general model of resolution. Then we shall define the notion of a resolution theory and we shall show that, for each CSP, a Basic Resolution Theory can be defined. Even though this Basic Theory may not be very powerful, it will be the basis for defining more elaborate ones; it is therefore “basic” in the two meanings of the word.

1.3. Parameters and instances of a CSP; minimal instances; classification

Generally, rather than a single problem, a CSP defines a whole family of problems. Typically, there is an integer parameter that splits this family into subclasses. A good example of such a parameter is the size of the grid in n-Queens, Latin Squares or Sudoku. In the resource allocation problem, it could be some combination of the number of resources and the number of tasks competing for them. In graph colouring and graph matching, it can be the size of the graph (some combination of the number of nodes and the number of arcs).

1.3.1. Minimal instances

Typically also, once this main parameter has been fixed, there remains a whole family of instances of the CSP. In 9×9 Sudoku, an instance is defined by a set of givens. In n-Queens, although the usual presentation of the problem starts from an empty grid and asks for all the solutions, we shall adopt for our purposes another view of this CSP; it consists of setting a few initial entries and asking for a solution or a “readable” proof that there is none. In graph colouring, the possibilities are still more open: there may be lots of graphs of a given size and, once such a graph has been chosen, it may also be required to have predefined colours for some subsets of nodes. The same remarks apply to graph matching, where one may want to have predefined correspondences between some nodes of the two graphs.

In such cases, classifying all the instances of a CSP or doing statistics on the difficulty of solving them meets problems of two kinds. Firstly, lots of instances will have very easy solutions: if givens are progressively added to an instance, until only the values of few variables remain non given, the problem becomes easier and easier to solve. Conversely, if there are so few instances that the problem has several solutions, some of these may be much easier to find than others. These two types of situations make statistics on all the instances somewhat irrelevant. This is the motivation for the following definition (inherited from the Sudoku classics).

An instance of a CSP is called *minimal* if it has one and only one solution and any instance obtained from it by eliminating any of its givens has more than one solution.

For the above-mentioned reasons, all our statistical analyses of a CSP will be restricted to the set of its minimal instances.

1.3.2. Rating and the complexity distribution of instances

Classically, the complexity of a CSP is studied with respect to its main size parameter and one relies on a worst case (or more rarely on a mean case) analysis. It often reaches conclusions such as “this CSP is NP-complete” – as is the case for Sudoku(n) or Latin-Square(n), considered as depending on grid size.

The questions about complexity that we shall tackle in this book are of a very different kind; they will not be based on the main size parameter. Instead, they will be about the statistical complexity distribution of instances of a fixed size CSP.

This supposes that we define a measure of complexity for instances of a CSP. We shall therefore introduce several ratings (starting in chapter 5) that are meaningful for the general CSP. And we shall be able to give detailed results (in chapter 6) for the standard (i.e. 9×9) Sudoku case. In trying to do so, the problem arises of creating unbiased samples of minimal instances and it appears to be very much harder than one may expect. We shall be able to show this in full detail only for the particular Sudoku case, but our approach is sufficiently general to suggest that the same kind of problem is very likely to arise in any CSP.

Indeed, what we shall define is different measures of complexity for various families of resolution rules. For each of these families, the complexity of a CSP instance will be defined as the complexity of the hardest rule in this family necessary to solve it. This is *a priori* very different from a definition based on the full resolution path. Of course, there must be some relationship between a ranking based on the hardest step of the resolution paths and a ranking of the full resolution paths; but, given a fixed set of rules, Sudoku examples show that it can solve puzzles whose resolution paths vary largely in complexity (whatever intuitive notion of complexity one adopts for the paths). In this book however, we shall not deal with the very hard (and as yet untouched) problem of ranking the instances according to their full resolution path; we shall only consider the hardest step.

1.4. The basic and the more complex resolution theories of a CSP

Following the definition of the CSP graph in section 1.1.1, we say that two candidates are linked by a direct contradiction, or simply *linked*, if there is a constraint making them incompatible (including the "strong" constraints, usually not explicitly stated as such, that different values for a CSP variable are incompatible).

1.4.1. Universal elementary resolution rules and their limitations

Every CSP has a Basic Resolution Theory: BRT(CSP).

The simplest elimination rule (obviously valid for any CSP) is the direct translation of the initial problem formulation into operational rules for managing candidates. We call it the "elementary constraints propagation rule" (ECP):

– ECP: if a value is asserted for a CSP variable (as is the case for the givens), then remove any candidate that is linked to this value by a direct contradiction.

The simplest assertion rule (also obviously valid) is called Single (S):

– S: if a CSP variable has only one candidate left, then assert it as the only possible value of this variable.

There is also an obvious Contradiction Detection rule (CD):

– CD: if a CSP variable has no decided value and no candidate left, then conclude that the problem has no solution.

The "elementary rules" ECP, S and CD constitute the Basic Resolution Theory of the CSP, BRT(CSP).

In Sudoku, novice players may think that these three elementary rules express the whole problem and that applying them repeatedly is therefore enough to solve any puzzle. If such were the case, one would probably never have heard of Sudoku, because it would amount to mere paper scratching and it would soon become boring. Anyway, as they get stuck in situations in which they cannot apply any of these rules, they soon discover that, except for the simplest puzzles, this is very far from being sufficient. The puzzle in Figure 1.1 is a very simple illustration of how one gets stuck if one only knows and uses the elementary rules: the resulting situation is shown in Figure 1.2, in which none of these rules can be applied. For this puzzle, modelling considerations related to symmetry (chapter 2) lead to "Hidden Single" rules allowing to solve it, but even this is generally very far from being enough.

1.4.2. Derived constraints and more complex resolution theories

As we shall see later, there are lots of puzzles that need resolution rules of a much higher complexity than those in the Basic Sudoku Resolution Theory in order to be solved. And this is why this game has become so popular: all but the easiest puzzles require a particular combination of neuron-titillating techniques and they may even suggest the discovery of as yet unknown ones.

The general reason for the limited resolution power of the Basic Resolution Theory of any CSP can be explained as follows. Given a set of constraints, there are usually many "derived" or "implied" constraints not immediately obvious from the original ones. Resolution rules can be considered as a way of expliciting some of them. As we shall see that very complex resolution rules are needed to solve some instances of a CSP, this will show not only that derived constraints cannot be reduced to the elementary rules of the Basic Resolution Theory (which constitute a straightforward operationalization of the axioms) but also that they can be unimaginably more complex than the initial constraints.

With all our Sudoku examples being minimal instances, secondary questions about multiple or inexistent solutions are discarded. From an epistemological point of view, the gap between the *what* (the initial constraints) and the *how* (the resolution rules necessary to solve an instance) is thus exhibited in all its purity, in a concrete way understandable by anyone. I may be a very naive person, but I must confess that, in spite of my pure logic background and of my familiarity with all the well-known mathematical ideas more or less related to it (culminating in deterministic chaos), this gap has always been for me a subject of much wonder. It is undoubtedly one of the main reasons why I kept interested in the Sudoku CSP for much longer than I expected when I first chose it as a topic for practical classes in AI. All the families of resolution rules defined in this book can be seen as different ways of exploring this gap.

1.4.3. Resolution rules and resolution strategies; the confluence property

One last point can now be clarified: the difference between a resolution theory (a set of resolution rules) and a resolution strategy. Everywhere in this book, a *resolution strategy* must be understood in the following extra-logical sense:

– a set of *resolution rules*, i.e. a *resolution theory*;

– a *non-strict precedence ordering* of these rules. Non-strict means that two rules can have the same precedence (for instance, in Sudoku, there is no reason to give a rule higher precedence than that obtained from it by transposing rows and columns or by any of the generalised symmetries explained in chapter 2).

As a consequence of this definition, several resolution strategies can be based on the same resolution theory with different partial orderings and they may lead to different resolution paths for a given instance.

Moreover, with every resolution strategy one can associate several deterministic procedures for solving instances:
As a preamble (each of the following choices will generate a different procedure):
- list all the resolution rules in a way compatible with their precedence ordering (i.e. among the different possibilities of doing so, choose one);
- list all the labels in a predefined order;
Given an instance P, loop until a solution of P is found (or until all the solutions are found or until it is proven that P has no solution):

```
|  Do until a rule can effectively be applied:
|  |  Take the first rule not yet tried in the list
|  |  Do until its condition pattern is effectively active:
|  |  |  Try to apply all the possible mappings of the condition pattern of this rule
|  |  |  to subsets of labels, according to their order in the list of labels
|  |  End do
|  End do
|  Apply the rule to the selected matching pattern
End loop
```

In this context, a natural question arises: given a resolution theory T, can different resolution procedures built on T lead to an instance being finally solved by some of them and unsolved by others? The answer lies in the *confluence property* of a resolution theory, to be explained in chapter 5; this fundamental property implies that the order in which the rules of T are applied is irrelevant as long as we are only interested in solving instances (but it can still be relevant when we also consider the efficiency of the procedure): all the resolution paths will lead to the same final state.

This apparently abstract confluence property (first introduced in *HLS1*) has very practical consequences when it holds in a resolution theory T. It allows any opportunistic strategy, such as applying a rule as soon as a pattern instantiating it is found (e.g. instead of waiting to have found all the potential instantiations of rules of the same precedence before choosing which should be applied first). Most importantly, it also allows to define a "simplest first" strategy that is guaranteed to produce a correct rating of an instance with respect to T after following a single resolution path (with the easy to imagine computational consequences).

1.5. The roles of logic, AI, Sudoku and other examples

As its organisation shows, this book about the general CSP problem has a large part (one third) dedicated to illustrating the abstract concepts with a detailed case study of Sudoku and (to a much lesser extent) with examples from n-Queens and n-SudoQueens. Nevertheless, it can also be considered as a long exercise in either logic or AI. Let us clarify the roles we grant each of these disciplines.

1.5.1. The role of logic

Throughout this book, the main function of logic will be to provide a rigorous framework for the precise definition of our basic concepts (such as a "candidate", a "resolution rule" and a "resolution theory"). Apart from the formalisation of the CSP itself, the simplest and most striking example is the formalisation (in section 4.3) of the CSP Basic Resolution Theory introduced above, and of all the forthcoming more complex resolution theories. Logic will also be used as a compact notation tool for expressing some resolution rules in a non-ambiguous way. In the Sudoku example, it will also be a very useful tool for expliciting the precise symmetry relationships between different "Subset rules" (in chapter 8).

For better readability, the rules we introduce are always formulated first in plain English and their validity is only established by elementary non-formal means. The non-mathematically oriented reader should thus not be discouraged by the logical formalism. Moreover, for all the types of chains we shall consider, the associated rules will always be represented in a very intuitive, almost graphical formalism.

As a fundamental and practical application of our strict logical foundations to the Sudoku CSP, its natural symmetry properties can be transposed into three formal meta-theorems allowing one to deduce systematically new rules from given ones (see chapter 2 and sections 3.6 and 4.7). In *HLS*, this allowed us to introduce chain rules of completely new types (e.g. "hidden chains"). It also allowed to state a clear relationship between Sudoku and Latin Squares.

Finally, the other role assigned to logic is that of a mediator between the intuitive formulation of the resolution rules and their implementation in an AI program (e.g. our SudoRules solver). This is a methodological point for AI (or software engineering in general): no program development should ever be started before precise definitions of its components are given (though not necessarily in strict logical form) – a common sense principle that is very often violated, especially by those who consider it as obvious (this is the teacher speaking!). Notice however that the logical formalism is only one among other preliminaries to implementation and that it does not dispense with the need for some design work (be it only for efficiency matters!).

1.5.2. The role of AI

The role we impart to AI in this book is mainly that of providing a quick testbed for the general ideas developed in the theoretical part. The main rules have been implemented in our SudoRules solver. It was initially designed for Sudoku only (whence the name) and its input and output functions remain dedicated to Sudoku (and Latin Squares), but its heart (the rules themselves) is fully general and could be applied unchanged to any CSP (except the "Subset rules" in chapter 8, for which we have kept a Sudoku specific implementation, for efficiency reasons).

One important facet of the rules introduced in this book is their resolution power. This can only be tested on specific examples and Sudoku has served as our testbed for this. The resolution of each puzzle by a human solver needs a significant amount of time and the number of puzzles that can be tested "by hand" against any resolution method is very limited. On the contrary, implementing our resolution rules in a solver allowed us to test about ten millions of puzzles (see chapter 6). This also gave us indications of the relative efficiency of different rules. It is not mere chance that the writing of *HLS* and of the present book occurred in parallel with successive versions of SudoRules. Abstract definitions of the relative complexities of rules were checked against our puzzle collections for their resolution times and for their memory requirements (in terms of the number of partial chains generated).

This book can also be considered as the basis for a long exercise in AI. Many computer science departments in universities have used Sudoku for various projects. According to our personal experience, it is a most welcome topic for student projects in computer science or AI. Trying to implement some rules, even the "simple" Subset rules of chapter 8, shows how re-ordering the conditions can drastically change the behaviour of a knowledge-based system: without care, Quads easily lead to memory overflow problems. (We give detailed formulations for Subset rules in Sudoku, so that they can be used for such exercises without too long preliminaries.) Trying to implement S_p-whips or W_p-whips is a real challenge.

1.5.3. The role of Sudoku

Because some parts of this book related to the general CSP are rather abstract (e.g. chapters 3 and 4) or technical (e.g. chapters 9 and 10), a detailed case study was needed to show how the general concepts work in practice. It is also necessary to show how the general theory can easily be adapted, in the most important initial modelling phase, for dealing more efficiently or more naturally with each specific case. Choosing Sudoku for these purposes was for us a natural consequence of the historic development of the techniques described here, both the general approach and all the types of resolution rules. But there are many other reasons why it is an excellent example for the general CSP.

As can be seen from a fast browsing of this book, examples from the Sudoku CSP appear in many chapters (generally at the end, in order not to overload the main text with long resolution paths) and we keep our *HLS* constraint that all of them should originate in a real minimal puzzle – instead of an artificially created partial situation (resolution state).

But it should be clear for the readers of *HLS* that the purpose here is very different: we have no goal of illustrating with a Sudoku example each of the rules we introduce (for this, there is *HLS* or our website). Instead, each example satisfies a precise function with respect to the general Constraint Satisfaction Problem, such as providing a counter-example to some conjecture. As a result, most of our Sudoku examples will be exceptional cases, with very long resolution paths – which could give (without this warning) a very bad idea of how difficult the resolution paths look for the vast majority of puzzles; the statistics in chapter 6 will give a better idea: most of the time, the chains used and the paths are short.

1.5.3.1. Why Sudoku is a good example

Sudoku is known to be NP-complete [Gary & al. 1979]; more precisely, the CSP family Sudoku(n) on square grids of sizes n×n for all n is NP-complete. As we fix n = 9, this should not have any impact on our analyses. But the Sudoku case will exemplify very clearly (in chapter 6) that, for fixed n, the instances of an NP-complete problem often have a broad spectrum of complexity. It also shows that standard analyses, only based on worst case (worst instances), can be very far from reflecting the realities of a CSP.

For fixed n = 9, Sudoku is much easier to study than other readily formalised problems such as Chess or Go or any "real world" example. But it keeps enough structure so that it is not obvious.

Sudoku is a particular case of Latin Squares. Latin Squares are more elegant (and somehow more respectable) from a mathematical point of view, because they enjoy a complete symmetry of all the variables: numbers, rows, columns. In Sudoku, the constraint on blocks introduces some apparently mild complexity that makes it more exciting for players. But this lack of full symmetry also makes it more interesting from a theoretical point of view. In particular, it allows to introduce the notion of a grouped label and to identify a kind of general pattern based on it which is not present in Latin Squares: g-labels and the corresponding g-whips and g-braids (see chapter 7).

There are millions of Sudoku players all around the world and many forums where the rules defined in *HLS* have been the topic of much debate. A huge amount of invaluable experience has been cumulated and is readily available – including generators of random (but biased) puzzles, collections of puzzles with very specific properties (fish patterns, symmetry properties, …) and other collections of extremely hard puzzles. The lack of similar collections and of generators of minimal instances is a strong limitation for the detailed analysis of other CSPs.

1.5.3.2. Origin of our Sudoku examples

Most of our Sudoku examples rely on the following sets of minimal puzzles:

– the Sudogen0 collection consists of 1,000,000 puzzles randomly generated with the top-down suexg generator (http://magictour.free.fr/suexco.txt), with seed 0 for the random numbers generator; puzzle number n is named Sudogen0#n;

– the cb collection consists of 5,926,343 puzzles produced by a new kind of generator, the controlled-bias generator (we first introduced it on the Sudoku Player's Forum; see also [Berthier 2009] and chapter 6 below); it is still biased (though less than the previous ones) but in a known way, so that it allows to compute unbiased statistics; puzzle number n is named cb#n;

– the Magictour collection of 1465 puzzles considered to be the hardest (at the time of its publishing); puzzle number n is named Magictour-top1465#n;

– the gsf collection of 8152 puzzles considered to be the hardest (at the time of its publishing); puzzle number n is named gsf-top8152 #n.

Occasionally, we shall also use other puzzles from collections with particular properties, as specified in the text.

1.5.4. The role of non Sudoku examples

Although Sudoku is a very good CSP example, it has a few specificities, such as (the major one of) having all its constraints defined by CSP variables. The role of our other examples (Latin Squares, n-Queens, n-SudoQueens) is limited to reviewing a few of these specificities and to showing that they have no negative impact on our general theory, in particular that our main resolution rules (for whips, g-whips, Subsets, S_p-whips, W-whips, braids, …) can effectively be applied to other CSPs and how different these patterns may look in these cases.

We are aware that more examples should be given as much consideration as we have granted Sudoku and it is a shortcoming of this book that we did not do it; but time is not extensible and we hope that the approach described in this book will motivate more research for applications to other CSPs.

1.5.5. Uniform presentation of all the examples

If we displayed the full resolution path of an instance, it would generally take several pages, most of which would describe obvious or uninteresting steps. We shall skip most of these steps, with the following conventions (same as in *HLS*):

– elementary constraint propagation rules (ECP) will never be displayed;

– as the final rules that apply to any instance are always ECP and Singles (at least when these rules are given higher priority than more complex ones), they will be omitted from the end of the path.

All our examples respect the following uniform format. After an introductory text explaining the purpose of the example, the resolution theory T applied to it and/or comments on some particular point, a row of two (sometimes three) grids is displayed: the original puzzle (sometimes an intermediate state) and its solution. Then comes *the resolution path, a proof of the solution within theory T, where "proof" is meant in the strict sense of constructive logic*.

Each line in the resolution path consists of the name of the rule applied, followed by: the description of how the rule is "instantiated" (i.e. how the condition part is satisfied), the "==>" sign, the conclusion allowed by the "action" part. The conclusion is always either that a candidate can be eliminated (symbolically written as r4c8 ≠ 6 in Sudoku) or that a value must be asserted (symbolically written as r4c8 = 5). When the same rule instantiation justifies several conclusions, they are written on the same line, separated by commas: e.g. r4c8 ≠ 8, r5c8 ≠ 8. Occasionally, the detailed situation at some point in the resolution path (the "resolution state") is displayed so that the presence of the pattern under discussion can be directly checked, but, due to place constraints, this cannot be systematic.

For Sudoku, all the resolution paths given in this book were obtained with versions 13 or 15b of our SudoRules solver (with some occasional hand editing for a shorter and cleaner appearance), using the CLIPS inference engine (release 6.30), on a MacPro™ 2006 running at 2.66 GHz.

1.6. Notations

Throughout this book, we consider an arbitrary, but fixed, finite Constraint Satisfaction Problem. We call it CSP, generically. BRT(CSP) refers to its Basic Resolution Theory, RT to any of its resolution theories, W_n [respectively B_n, gW_n, gB_n, S_pW_n, S_pB_n, …] to its nth whip [respectively braid, g-whip, g-braid, S_p-whip, S_p-braid, …] resolution theory. The same letters, with no index, are used for the associated ratings. We use BSRT as an abbreviation for BRT(Sudoku) and BLSRT for BRT(LatinSquare).